



A Comparative Analysis of Convergence Techniques for the Homotopy Perturbation Method to Solve the Burgers' Equation in One Dimension

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تحليل مقارنة لتقنيات تسريع التقارب لطريقة الاضطراب الهوموتوبي لحل معادلة برجر في بعد واحد

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Abstract

This paper aims to select suitable initial approximations that increase the convergence rate and reduce the size of the accounts for the homotopy perturbation method (HPM). The study was conducted using two methods: the homotopy separation of variables (HSV) and the modified homotopy perturbation method (MHPM). The methods were tested on three problems of the Burgers' equation. Although the modified methods had the same number of iterations, the results showed that, in one example, the magnitude of error in the MHPM was significantly lower. Therefore, the MHPM is more effective than the HSV. Thus, the selection of the optimal initial approximation has an impact on improving the performance of the HPM.

Keywords: Homotopy separation of Variables, Modified Homotopy Perturbation Method, Burger's Equation, Nonlinear partial differential equations.

المخلص

تهدف هذه الورقة إلى اختيار تقريبات أولية ملائمة تزيد من معدل التقارب وتقلل من حجم العمليات الحسابية المصاحبة لطريقة الاضطراب الهوموتوبي. تمت الدراسة بطريقتين: طريقة الفصل الهوموتوبي للمتغيرات وطريقة الاضطراب الهوموتوبي المعدلة.



اختبرت الطرق على ثلاثة مسائل لمعادلة برجر التفاضلية الجزئية غير الخطية. وبالرغم من أن الطرق المعدلة لها نفس عدد التكرارات إلا أن النتائج أظهرت في أحد الأمثلة أن مقدار الخطأ في MHPM أقل بكثير، لذا فإن طريقة الاضطراب الهوموتوبي المعدلة أكثر فاعلية من طريقة الفصل الهوموتوبي للمتغيرات، وبالتالي فإن اختيار التقريب الابتدائي الأمثل له تأثير على تحسين أداء طريقة الاضطراب الهوموتوبي.

الكلمات الدالة: طريقة الفصل الهوموتوبي للمتغيرات، طريقة الاضطراب الهوموتوبي المعدلة، معادلة برجر، المعادلات التفاضلية الجزئية غير الخطية.

1. Introduction

Partial differential equations are fundamental mathematical tools that arise in many physical and engineering phenomena; among them, the Burgers' equation models fluid dynamics. Many authors have studied this equation to understand physical flows or to test numerical methods because its properties combine nonlinearity with relative simplicity. The general form of this equation is (Burgers, 1940):

$$\frac{\partial \omega}{\partial s} + \omega \frac{\partial \omega}{\partial x_1} = \nu \frac{\partial^2 \omega}{\partial x_1^2} \quad : 0 < x_1 < a, s \geq 0 \quad \dots (1)$$

Where: ω the velocity, x_1 is position, s is time, and ν the Kinematic viscosity.

Bateman (1915) introduced Burger's equation and identified a special solution of specific viscous flows. Burger (1948) formulated this equation to describe certain characteristics of turbulent fluid; thus, it is often called Burger's equation. Hopf (1915) and Cole (1951) solved the Burgers equation with arbitrary initial conditions.

In recent years, researchers have focused on the Homotopy Perturbation Method (HPM) and based techniques on it, adapting them to address increasingly complex mathematical and physical problems. This method was introduced by He (1999). The fundamental principle of the HPM involves creating a homotopy, which is a continuous transformation or path that links an easily solvable equation to the equation that needs to be solved. Researchers found that the complexity of the calculations in HPM depends on the initial approximation chosen (Yang et al., 2012). Biazar and Ghazvini (2009) proved the convergence of this method in solving PDEs with a suitable initial condition.



2. Literature Review

The following review highlights the main developments related to the use of this method in solving Burgers' equations;

The authors (Ganji et al., 2010; Rathore & Saraswat, 2018; Singh et al., 2020; Kapoor, 2020) conducted numerical analyses of HPM on Burgers' equations to verify the effectiveness of the method among them.

Elzaki & Kim (2014), Hamza & Elzaki (2015), and Olubanwo et al. (2019) investigated the possibility of increasing the efficiency of HPM by combining an integral transformation with HPM. They found that this method reduces computation for HPM.

Mirzazadeh & Ayati (2016) and Nandhini & Sumathi (2024) used a new effective approach, known as the modified homotopy perturbation method (MHPM). This method is based on and assuming that the initial approximation is in the form of a function series and that the subsequent approximations vanish.

In this paper, we present MHPM and another method known as the homotopy separation of variables (HSV): This method, taken from papers by Yang et al. (2012) and Karbalaie & Muhammad (2014), involves assuming a separation of variables in the initial approximation and using this to estimate the solution, and assuming the subsequent approximations vanish.

3. Methodology

3.1. The core principle of the HPM

To explain the core principle of the HPM, let's study the following nonlinear differential equation:

$$\mathcal{F}(\omega(x)) - h(x) = 0, \quad x \in \Omega \quad \dots (2)$$

With the boundary conditions;

$$\mathcal{B}\left(\omega(x), \frac{\partial \omega(x)}{\partial n}\right) = 0, \quad x \in \eta \quad \dots (3)$$



$\mathcal{F}$ is a differential operator can be separated into two parts $\mathcal{L}$ and $\mathfrak{N}$, where $\mathcal{L}$ is linear operator and $\mathfrak{N}$ is nonlinear, $\mathcal{B}$ is a boundary operator, $h(x)$ is an analytic function, and η is the domain boundary Ψ . The Eqn. (2) can be rewritten as follows:

$$\mathcal{M}(\omega(x)) + \mathfrak{N}(\omega(x)) - h(x) = 0 \quad \dots (4)$$

Using the Homotopy technique, we construct a homotopy $\zeta(x, q): \Psi \times [0, 1] \rightarrow \mathfrak{R}$, that fulfills:

$$\mathcal{H}(\zeta(x), q) = (1 - q)[\mathcal{M}(\zeta(x)) - \mathcal{M}(\omega_0(x))] + q[\mathcal{F}(\zeta(x)) - h(x)] = 0 \quad \dots (5)$$

$$\text{or } \mathcal{H}(\zeta, q) = R(\zeta(x)) - R(\omega_0(x)) + q\omega_0(x) + q[\mathfrak{N}(\zeta(x)) - h(x)] = 0 \quad \dots (6)$$

Where $q \in [0, 1]$ represents an embedding parameter, and ω_0 serves as an initial estimate of Eqn. (2) that meets the boundary conditions outlined in Eqn. (3). From Eqn. (5), we derive;

$$\mathcal{H}(\omega(x), 0) = \mathcal{M}(\zeta(x)) - \mathfrak{N}(\omega_0(x)) = 0 \quad \dots (7)$$

$$\mathcal{H}(\omega(x), 1) = \mathcal{F}(\zeta(x)) - h(x) = 0 \quad \dots (8)$$

When changing q from zero to unity, $\zeta(x, q)$ changing from $\omega_0(x)$ to $\omega(x)$, and Eqns. (7) and (8) are called homotopic in topology.

Now, by perturbation technique, assume that the solution of Eqn. (5) can be represented as a series in q :

$$\zeta(x) = \sum_{i=0}^{\infty} q^i \zeta_i(x) \quad \dots (9)$$

When $q \rightarrow 1$: Eqn. (5) is equal to Eqn. (2) i.e. an follows the equation becomes an approximate solution of Eqn. (2) (Biazar and Ghazvini, 2009);

$$\omega(x) = \lim_{q \rightarrow 1} \zeta(x) = \sum_{i=0}^{\infty} \zeta_i(x) \quad \dots (10)$$

3.2. Adapting the HPM to Solve the Burger's Equation

Let consider the Burger's equation;

$$\frac{\partial \omega}{\partial s} + \omega \frac{\partial \omega}{\partial x_1} = \nu \frac{\partial^2 \omega}{\partial x_1^2} \quad : 0 < x_i < a, s \geq 0 \quad \dots (11)$$

Subject to the conditions;

$$\omega(x_1, 0) = h_0(x_1) \quad \dots (12)$$

where $\mathfrak{N} = \omega \frac{\partial \omega}{\partial x_1}$. To solve Eqn. (11) with condition (12), we can construct homotopy;



$$(1 - q) \left[\frac{\partial \zeta}{\partial s} - \frac{\partial \omega_0}{\partial s} \right] + q \left[\frac{\partial \zeta}{\partial s} + \zeta \frac{\partial \zeta}{\partial x_1} - \nu \frac{\partial^2 \zeta}{\partial x_1^2} \right] = 0 \quad \dots (13)$$

Alternatively;

$$\frac{\partial \zeta}{\partial s} - \frac{\partial \omega_0}{\partial s} = q \left[\nu \frac{\partial^2 \zeta}{\partial x_1^2} - \frac{\partial \omega_0}{\partial s} - \zeta \frac{\partial \zeta}{\partial x_1} \right] \quad \dots (14)$$

Substituting Eqn. (9) into Eqn. (14) and equating the coefficients of like powers of q , we obtain;

$$\begin{aligned} q^0: \frac{\partial \zeta_0}{\partial s} - \frac{\partial \omega_0}{\partial s} &= 0, \quad \zeta_0(x_1, 0) = \omega_0(x_1, 0) \\ q^1: \frac{\partial \zeta_1}{\partial s} + \frac{\partial \zeta_0}{\partial s} + \nu_0 \frac{\partial \zeta_0}{\partial x_1} - \nu \frac{\partial^2 \zeta_0}{\partial x_1^2} &= 0, \quad \zeta_1(x_1, 0) = 0 \\ q^2: \frac{\partial \zeta_2}{\partial s} + \zeta_0 \frac{\partial \zeta_1}{\partial x_1} + \zeta_1 \frac{\partial \zeta_0}{\partial x_1} - \nu \frac{\partial^2 \zeta_1}{\partial x_1^2} &= 0, \quad \zeta_2(x_1, 0) = 0 \\ &\vdots \\ q^j: \frac{\partial \zeta_j}{\partial s} + \sum_{k=0}^{j-1} \zeta_k \frac{\partial \zeta_{j-k-1}}{\partial x_1} - \nu \frac{\partial^2 \zeta_{j-1}}{\partial x_1^2} &= 0, \quad \zeta_j(x_1, 0) = 0 \end{aligned} \quad \dots (15)$$

we often take $\zeta_0(x_1, t) = \omega_0(x_1, t) = h_0(x)$.the solution is (Kapoor, 2020);

$$\omega(x) = \sum_{i=0}^{\infty} \zeta_i(x) \quad \dots (16)$$

3.3. The Basic Idea of the HSV

We start first assume that the variables of $\omega_0(x_1, t)$ can be separated as:

$$\omega_0(x_1, t) = \omega(x_1, 0)\alpha_1(s) + \frac{\partial \zeta_0(x_1, 0)}{\partial x_1} \alpha_2(s) \quad \dots (17)$$

and assume that $\zeta_1 \equiv 0$ in Eqn. (15) which means that $\zeta_n \equiv 0, \forall n \geq 1$. Then, the solution of Eqn. (11) is

$$\zeta(x_1, t) = \zeta_0(x_1, t) = \omega_0(x_1, t) \quad \dots (18)$$

We get the solution by finding $\alpha_1(s)$ and $\alpha_2(s)$. To achieve this goal, we consider the condition on Eqn. (12):

$$\omega(x_1, 0) = h_0(x_1) \quad \dots (19)$$

Since Eqn. (18) satisfies the initial, we get (Yang et al., 2012; Karbalaie & Muhammed, 2014);

$$\alpha_1(0) = 1, \alpha_2(0) = 0 \quad \dots (20)$$

3.4. The Basic Idea of the MHPM

In this modified: We start construct the following homotopy;



$$\frac{\partial \zeta}{\partial s} = \frac{\partial \omega_0}{\partial s} + q \left[v \frac{\partial^2 \zeta_0}{\partial x_1^2} - \frac{\partial \omega_0}{\partial s} - \zeta \frac{\partial \zeta}{\partial x_1} \right] \quad \dots (21)$$

Where; $\zeta_0(x_1, 0) = \omega_0(x_1, 0)$

Let's present the solution of Eqn. (21) as follows;

$$\zeta(x_1) = \sum_{i=0}^{\infty} q^i \zeta_i(x) \quad \dots (22)$$

Suppose the initial approximation of Eqn. (21) is in the following form:

$$\zeta_0(x_1, s) = \varepsilon_0(x_1) + \sum_{m=0}^{\infty} \beta_m(x_1) \int_0^t \mathbb{Q}_m(t) dt \quad \dots (23)$$

Where $\beta_m, m = 0, 1, 2, \dots$ are unknown coefficients and $\mathbb{Q}_m(t), m = 0, 1, 2, \dots$ are functions depending on the problem. Now by substituting Eqn. (23) and Eqn. (22) into the Eqn. (21), and equating the coefficients of q with the same powers. We get;

$$\begin{aligned} q^0: \zeta_0(x_1, s) &= \varepsilon_0(x_1) + \sum_{m=0}^{\infty} \beta_m \int_0^t \mathbb{Q}_m(t) dt \\ q^1: \zeta_1(x_1, s) &= \int_0^t \left(v \frac{\partial^2 \zeta_0}{\partial x_1^2} - \frac{\partial \zeta_0}{\partial s} - \zeta_0 \frac{\partial \zeta_0}{\partial x_1} \right) dt \\ &\vdots \\ q^j: \zeta_j(x_1, s) &= \int_0^t \left(v \frac{\partial^2 \zeta_{j-1}}{\partial x_1^2} - \sum_{k=0}^{j-1} \zeta_k \frac{\partial \zeta_{j-k-1}}{\partial x_1} \right) dt, \quad j = 2, 3, \dots \end{aligned} \quad \dots (24)$$

and so on...

Now if $\zeta_1 = 0$, then Eqn. (24) results in $\zeta_n = 0, n \geq 1$. Therefore; the analytic solution may be obtained as;

$$\omega(x, s) = \zeta_0(x_1, s) \quad \dots (25)$$

It is worthwhile to mention that if $\zeta_0(x_1, s)$ are analytic around $s = 0$, then their Taylor series can be defined as;

$$\zeta_0(x_1, s) = \sum_{m=0}^{\infty} \beta_m(x_1) s^m \quad \dots (26)$$

can be used in Eqn. (24) (Mirzazadeh & Ayati, 2016; Nandhini & Sumathi, 2024).

4. Experiment Comparison

Example (1): Consider the Burger's equation:

$$\frac{\partial \omega}{\partial s} + \omega \frac{\partial \omega}{\partial x_1} = v \frac{\partial^2 \omega}{\partial x_1^2} \quad \dots (27)$$

With the initial condition as:

$$\omega(x_1, 0) = x_1 \quad \dots (28)$$



The analytic solution of this problem is (Hamza and Elzaki, 2015); $\frac{x_1}{1+s}$

The solution by the HPM:

We construct a homotopy $\zeta: \Psi \times [0,1] \rightarrow \Re$, that satisfies:

$$(1-q) \left[\frac{\partial \zeta}{\partial s} - \frac{\partial \zeta_0}{\partial s} \right] + q \left[\frac{\partial \zeta}{\partial s} + \zeta \frac{\partial \zeta}{\partial x_1} - v \frac{\partial^2 \zeta}{\partial x_1^2} \right] = 0 \quad \dots (29)$$

Starting with the initial approximate

$$\zeta_0(x_1, s) = \omega_0(x_1, s) = \omega(x_1, 0)$$

And from the Eqn. (15). The solution is;

$$\frac{\partial \zeta_1}{\partial s} + 0 + x_1(1) - 0 = 0 \Rightarrow \zeta_1(x_1, s) = -x_1 s$$

$$\frac{\partial \zeta_2}{\partial s} + x_1(-s) + (-x_1 s)(1) - 0 = 0 \Rightarrow \zeta_2(x_1, s) = x_1 s^2$$

$$\frac{\partial \zeta_3}{\partial s} + x_1(s^2) + (-x_1 s)(-s) - x_1 s^2(-1) = 0 \Rightarrow \zeta_3(x_1, s) = -x_1 s^3$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$\begin{aligned} \omega(x_1, s) &= x_1 - x_1 s + x_1 s^2 - x_1 s^3 + \dots \\ &= x_1(1 - s + s^2 - s^3 + \dots) \end{aligned}$$

Consequently, the analytic solution is given by:

$$\omega(x_1, s) = \frac{x_1}{1+s} \quad |s| < 1$$

The Solution By HSV:

By this method, assume that;

$$\omega_0(x_1, s) = \omega(x_1, 0)\alpha_1(s) + \frac{\partial \omega_0(x_1, 0)}{\partial x_1} \alpha_2(s) \quad \dots (30)$$

whereas $\omega(x_1, 0) = x_1$, therefore;

$$\omega_0(x_1, s) = x_1 \alpha_1(s) + \alpha_2(s)$$

And;

$$\frac{\partial \zeta_1}{\partial s} = \frac{\partial^2 \zeta_0}{\partial x_1^2} - \frac{\partial \omega_0}{\partial s} - \zeta_0 \frac{\partial \zeta_0}{\partial x_1}$$

$$\Rightarrow \frac{\partial \zeta_1}{\partial s} = 0 - [x_1 \alpha_1'(s) + \alpha_2'(s)] - [x_1 \alpha_1(s) + \alpha_2(s)](\alpha_1(s))$$

$$\Rightarrow \frac{\partial \zeta_1}{\partial s} = [\alpha_1'(s) + \alpha_1^2(s)]x_1 + \alpha_2'(s) + \alpha_1(s)\alpha_2(s) \quad \dots (31)$$

By vanishing v_1 , We obtain the differential system as follows:



$$\alpha_1'(s) + \alpha_1^2(s) = 0, \alpha_1(0) = 1 \quad \dots (32)$$

$$\alpha_2'(s) + \alpha_1(s)\alpha_2(s) = 0, \alpha_2(0) = 0 \quad \dots (33)$$

Solving this system, we have;

$$\alpha_1(s) = \frac{1}{1+s}, \alpha_2(s) = 0 \quad \dots (34)$$

The analytic solution is then $\omega(x_1, s) = \frac{x_1}{1+s}$

The solution by MHPM

Assuming;

$$\omega_0(x_1, s) = h_0(x_1) + \sum_{m=0}^{\infty} \beta_m \int_0^s \mathbb{Q}_m(s) ds \quad \dots (35)$$

whereas $\omega(x_1, 0) = x_1$, therefore;

$$\zeta_0(x_1, s) = \omega_0(x_1, s) = x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots$$

And

$$\begin{aligned} \frac{\partial \zeta_1}{\partial s} &= \frac{\partial^2 \zeta_0}{\partial x_1^2} - \frac{\partial \zeta_0}{\partial s} - \zeta_0 \frac{\partial \zeta_0}{\partial x_1} \\ \frac{\partial \zeta_1}{\partial s} &= \left(\beta_0'' s + \frac{1}{2} \beta_1'' s^2 + \frac{1}{3} \beta_2'' s^3 + \dots \right) - \left(x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots \right) \left(1 + \beta_0' s + \frac{1}{2} \beta_1' s^2 + \frac{1}{3} \beta_2' s^3 + \dots \right) - (\beta_0 + \beta_1 s + \beta_2 s^2 + \dots) \\ \frac{\partial \zeta_1}{\partial s} &= (-x_1 - \beta_0) + (\beta_0'' - \beta_0 - x_1 \beta_0' - \beta_1) s + \left(\frac{1}{2} \beta_1'' - \frac{1}{2} \beta_1' x_1 - \beta_0 \beta_0' - \frac{1}{2} \beta_1 - \beta_2 \right) s^2 + \dots \end{aligned} \quad \dots (36)$$

By Vanishing ζ_1 the coefficients $\beta_m(x)$ are determined as:

$$\beta_0(x_1) = -x_1, \beta_1(x_1) = 2x_1, \beta_2(x_1) = -3x_1$$

Therefore, we gain the solution;

$$\begin{aligned} \zeta(x_1, s) &= \zeta_0(x_1, s) = x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots \\ &= x_1 - x_1 s + x_1 s^2 - x_1 s^3 + \dots \\ &= x_1 (1 - s + s^2 - s^3 + \dots) \end{aligned}$$

Consequently, the analytic solution is given by:

$$\omega(x_1, s) = \frac{x_1}{1+s} \quad |s| < 1$$

Example (2): Consider Burgers equation with the initial condition as:

$$u(x_1, 0) = 1 - \frac{2}{x_1} \quad \dots (37)$$



The analytic solution of this problem is $\omega(x_1, s) = 1 - \frac{2}{x_1 - s}$ (Hamza and Elzaki, 2015).

The solution by the HPM:

We construct a homotopy $\zeta: \Psi \times [0, 1] \rightarrow \Re$, that satisfies:

$$(1 - q) \left[\frac{\partial \zeta}{\partial s} - \frac{\partial \omega_0}{\partial s} \right] + q \left[\frac{\partial \zeta}{\partial s} + \zeta \frac{\partial \zeta}{\partial x_1} - \nu \frac{\partial^2 \zeta}{\partial x_1^2} \right] = 0$$

Starting with the initial approximate;

$$\zeta_0(x_1, s) = \omega_0(x_1, s) = \omega(x_1, 0)$$

And from the Eqn. (15). The solution is;

$$\zeta_1(x_1, s) = -\frac{2}{x_1^2} s, \quad \zeta_2(x_1, s) = -\frac{2}{x_1^3} s^2, \quad \zeta_3(x_1, s) = -\frac{2}{x_1^4} s^3$$

And so on. the solution in a series form is given by:

$$\begin{aligned} u(x, t) &= 1 - \frac{2}{x_1} - \frac{2}{x_1^2} s - \frac{2}{x_1^3} s^2 - \frac{2}{x_1^4} s^3 - \dots \\ &= 1 - \frac{2}{x_1} \left(1 + \frac{s}{x_1} + \frac{s^2}{x_1^2} + \frac{s^3}{x_1^3} + \dots \right) \end{aligned}$$

Consequently;

$$w(x_1, s) = 1 - \frac{2}{x_1 - s}$$

The Solution By HSV:

By this method, assume that;

$$\omega_0(x_1, s) = \omega(x_1, 0)\alpha_1(s) + \frac{\partial \omega_0(x_1, 0)}{\partial x_1} \alpha_2(s)$$

whereas $\omega(x_1, 0) = x_1$, therefore;

$$\omega_0(x_1, s) = \left(1 - \frac{2}{x_1} \right) \alpha_1(s) + \frac{2}{x_1^2} \alpha_2(s)$$

And

$$\begin{aligned} \frac{\partial \zeta_1}{\partial s} &= \frac{\partial^2 \omega_0}{\partial x_1^2} - \frac{\partial \omega_0}{\partial s} - \omega_0 \frac{\partial \omega_0}{\partial x_1} \\ \Rightarrow \frac{\partial \zeta_1}{\partial s} &= \left[-\frac{4}{x_1^3} \alpha_1(s) + \frac{12}{x_1^4} \alpha_2(s) \right] - \left[\left(1 - \frac{2}{x_1} \right) \alpha_1(s) + \frac{2}{x_1^2} \alpha_2(s) \right] \left[\frac{2}{x_1^2} \alpha_1(s) - \frac{4}{x_1^3} \alpha_2(s) \right] - \left[\left(1 - \frac{2}{x_1} \right) \alpha_1'(s) + \frac{2}{x_1^2} \alpha_2'(s) \right] \dots (38) \end{aligned}$$

By vanishing ζ_1 , and using the conditions $\alpha_1(0) = 1, \alpha_2(0) = 0$. we have;

$$\alpha_1(s) = 1, \alpha_2(s) = 0$$



Consequently $\omega(x_1, t) = 1 - \frac{2}{x_1}$

The solution by MHPM:

Assuming;

$$\omega_0(x_1, s) = h_0(x_1) + \sum_{m=0}^{\infty} \beta_m \int_0^s \mathbb{Q}_m(s) ds$$

whereas $\omega(x_1, 0) = 1 - \frac{2}{x_1}$, therefore;

$$\zeta_0(x_1, s) = \omega_0(x_1, s) = 1 - \frac{2}{x_1} + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots$$

And

$$\begin{aligned} \frac{\partial \zeta_1}{\partial s} &= \frac{\partial^2 \zeta_0}{\partial x_1^2} - \frac{\partial \zeta_0}{\partial s} - \zeta_0 \frac{\partial \zeta_0}{\partial x_1} \\ \frac{\partial \zeta_1}{\partial s} &= \left(-\frac{4}{x_1^3} + \beta_0'' s + \frac{1}{2} \beta_1'' s^2 + \dots \right) - \left(1 - \frac{2}{x_1} + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots \right) \left(\frac{2}{x_1^2} + \beta_0' s + \frac{1}{2} \beta_1' s^2 + \frac{1}{3} \beta_2' s^3 + \dots \right) - (\beta_0 + \beta_1 s + \beta_2 s^2 + \dots) \dots \quad (39) \end{aligned}$$

By Vanishing v_1 the coefficients $\beta_m(x_1)$ are determined as:

$$\beta_0(x_1) = -\frac{2}{x_1^2}, \quad \beta_1(x_1) = -\frac{4}{x_1^3}, \quad \beta_2(x_1) = -\frac{6}{x_1^4}$$

Therefore; we gain the solution;

$$\begin{aligned} \omega(x_1, s) &= \omega_0(x_1, s) = x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots \\ &= x_1 - \frac{2}{x_1^2} s - \frac{2}{x_1^3} s^2 - \frac{2}{x_1^4} s^3 - \dots \\ &= 1 - \frac{2}{x_1} \left(1 + \frac{s}{x_1} + \frac{s^2}{x_1^2} + \frac{s^3}{x_1^3} + \dots \right) \end{aligned}$$

Consequently;

$$\omega(x_1, s) = 1 - \frac{2}{x_1 - s}$$

From this example, we observe that the solution obtained from HVS is not convergent to the exact solution, because it does not depend on s . The divergence is also evident in Figures (1 and 2).

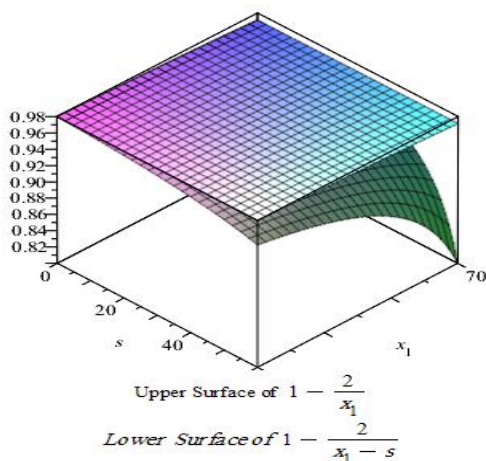


Figure 1. The comparison of the HSV and MHPM when solving Example (2)

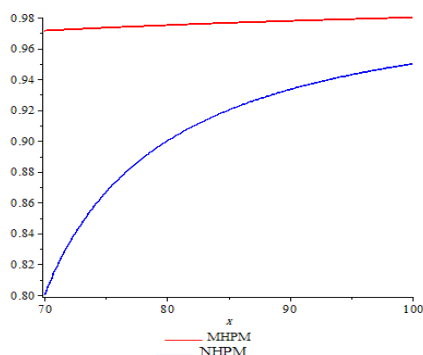


Figure 2. The comparison of the HSV and MHPM at S=60 when solving Example (2)

Example (3): Consider Burger's equation with the initial condition as:

$$\omega(x_1, 0) = 2 \tan x_1 \quad \dots (40)$$

The analytic solution of this problem is $\omega(x_1, s) = 2 \tan x_1$ (Hamza and Elzaki, 2015)

The solution by the HPM:

We construct a homotopy $\zeta: \Psi \times [0, 1] \rightarrow \Re$, that fulfills:

$$(1 - q) \left[\frac{\partial \zeta}{\partial s} - \frac{\partial \omega_0}{\partial s} \right] + q \left[\frac{\partial \zeta}{\partial s} + \zeta \frac{\partial \zeta}{\partial x_1} - \nu \frac{\partial^2 \zeta}{\partial x_1^2} \right] = 0$$

Starting with the initial approximate;

$$\zeta_0(x_1, s) = \omega_0(x_1, s) = \omega(x_1, 0)$$

and from the Eqn. (15). The solution is;



$$\zeta_1(x_1, s) = 0, \quad \zeta_2(x_1, s) = 0, \quad \zeta_3(x_1, s) = 0, \dots$$

and so on. Therefore;

$$\omega(x_1, s) = 2 \tan x_1$$

The Solution By HSV:

By this method, assume that;

$$\omega_0(x_1, s) = \omega(x_1, 0)\alpha_1(s) + \frac{\partial \omega_0(x_1, 0)}{\partial x_1} \alpha_2(s)$$

whereas $\omega(x_1, 0) = 2 \tan x_1$, therefore;

$$\omega_0(x_1, s) = 2 \tan x_1 \alpha_1(s) + 2 \sec^2 x_1 \alpha_2(s)$$

And

$$\begin{aligned} \frac{\partial \zeta_1}{\partial s} &= \frac{\partial^2 \omega_0}{\partial x_1^2} - \frac{\partial \omega_0}{\partial s} - \omega_0 \frac{\partial \omega_0}{\partial x_1} \\ \Rightarrow \frac{\partial \zeta_1}{\partial s} &= [4 \sec^2 x_1 \tan x_1 \alpha_1(s) + (4 \sec^4 x_1 + 8 \sec^2 x_1 \tan^2 x_1) \alpha_2(s)] - \\ &\quad [2 \tan x_1 \alpha_1(s) + 2 \sec^2 x_1 \alpha_2(s)][2 \sec^2 x_1 \alpha_1(s) + \\ &\quad 4 \sec^2 x_1 \tan x_1 \alpha_2(s)] - [2 \tan x_1 \alpha_1'(s) + 2 \sec^2 x_1 \alpha_2'(s)] \quad \dots (41) \end{aligned}$$

By vanishing ζ_1 , and using the conditions $\alpha_1(0) = 1, \alpha_2(0) = 0$. we have

$$\alpha_1(s) = 1, \alpha_2(s) = 0$$

$$\text{then } \omega(x_1, s) = 2 \tan x_1$$

The solution by MHPM:

Assuming;

$$\omega_0(x_1, s) = h_0(x_1) + \sum_{m=0}^{\infty} \beta_m \int_0^s \mathbb{Q}_m(s) ds$$

whereas $\omega(x_1, 0) = 2 \tan x_1$, therefore;

$$\zeta_0(x_1, s) = \omega_0(x_1, s) = 2 \tan x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots$$

And ;

$$\begin{aligned} \frac{\partial \zeta_1}{\partial s} &= \frac{\partial^2 \zeta_0}{\partial x_1^2} - \frac{\partial \zeta_0}{\partial s} - \zeta_0 \frac{\partial \zeta_0}{\partial x_1} \\ \frac{\partial \zeta_1}{\partial s} &= \left(4 \sec^2 x_1 \tan x_1 + \beta_0' s + \frac{1}{2} \beta_1'' s^2 + \dots \right) - \left(2 \tan x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \right. \\ &\quad \left. \frac{1}{3} \beta_2 s^3 + \dots \right) \left(2 \sec^2 x_1 + \beta_0' s + \frac{1}{2} \beta_1' s^2 + \frac{1}{3} \beta_2' s^3 + \dots \right) - (\beta_0 + \beta_1 s + \\ &\quad \beta_2 s^2 + \dots) \quad \dots (42) \end{aligned}$$



By Vanishing ζ_1 the coefficients $\beta_m(x_1)$ are determined as:

$$\beta_0(x_1) = 0, \quad \beta_1(x_1) = 0, \quad \beta_2(x_1) = 0$$

Therefore; we gain the solution;

$$\begin{aligned}\omega(x_1, s) = \omega_0(x_1, s) &= 2 \tan x_1 + \beta_0 s + \frac{1}{2} \beta_1 s^2 + \frac{1}{3} \beta_2 s^3 + \dots \\ &= 2 \tan x_1\end{aligned}$$

5. Conclusion

The study demonstrated that selecting the appropriate initial approximation in the HPM is crucial to achieve a balance between accuracy, convergence speed, and computational efficiency. One method: MHPM was used to accelerate convergence and reduce computational size while maintaining an accurate solution in the two studied examples, while the other method: HVS showed very fast convergence and a small computational size in one example but diverged in the other, highlighting the importance of avoiding methods that exhibit divergence to ensure stable performance. Therefore, we recommend relying on methods proven stable across multiple models and avoiding those that diverge to prevent loss of accuracy and computational efficiency. So, the proposed methods can be tested on other types of partial differential equations, such as the Schrodinger equation and the Berger equation in two dimensions.

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