



n-Magic Groups and Some New Directions

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الزمر السحرية من الرتبة n وبعض الاتجاهات البحثية الجديدة

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Abstract

Magic squares are classical mathematical objects characterized by elegant symmetry, where the sum of each row, column, and diagonal is constant. In this paper, we extend this notion from integers to algebraic structures by introducing n-magic groups. An n-magic group is defined as a finite or finitely generated group whose elements can be arranged in an $n \times n$ array such that the product (or sum, in the abelian case) of every row, column, and diagonal equals a fixed group element. We present constructions for both abelian and non-abelian groups, including $Z_n \times Z_n$ and $S_3 \times Z_3$, and show that the magic property is preserved under direct products. For finitely generated abelian groups, we give sufficient and necessary conditions for the existence of 3-magic arrays, emphasizing the role of subgroups isomorphic to $Z_3 \times Z_3$. In addition, we introduce complex n-magic squares based on roots of unity, highlighting connections between magic groups, polynomial roots on the unit circle, and complex multiplication. These results unify algebraic and analytic perspectives on magic structures and provide explicit constructions that open new directions for further research in algebra and discrete mathematics.

Keywords: Magic square, Magic group, Abelian group, non-abelian group, Complex magic square.

المخلص

تُعدّ المربعات السحرية من البنى الرياضية الكلاسيكية التي تتميز بتناظرٍ أنيق، حيث يكون مجموع عناصر كل صف وعمود وقطر ثابتاً. في هذا البحث، نُوسّع هذا المفهوم من نطاق الأعداد الصحيحة إلى البنى الجبرية من خلال تقديم مفهوم الزمر السحرية من



الرتبة (n-Magic Groups). وتُعرّف الزمرة السحرية من الرتبة n بأنها زمرة منتهية أو مولدة توليداً منتهاً يمكن ترتيب عناصرها في مصفوفة $n \times n$ بحيث يكون حاصل ضرب (أو مجموعها في حالة الزمر الإبدالية) عناصر كل صف وعمود وقطر مساوياً لعنصر ثابت في الزمرة. نقدّم في هذا العمل إنشاءات صريحة لكلٍ من الزمر الإبدالية وغير الإبدالية، بما في ذلك $S_3 \times Z_3$ و $Z_n \times Z_n$ ، ونُبرهن أن الخاصية السحرية تُحفظ تحت عملية الضرب المباشر للزمر. كما نعرض، بالنسبة للزمر الإبدالية المولدة توليداً منتهاً، شروطاً كافية وضرورية لوجود مصفوفات سحرية من الرتبة 3، مع إبراز الدور المحوري للزمر الجزئية المتشاكلية مع $Z_3 \times Z_3$. وإضافةً إلى ذلك، نستحدث مفهوم المربعات السحرية العقدية من الرتبة n المبنية على جذور الوحدة، مبرزين الروابط بين الزمر السحرية وجذور كثيرات الحدود الواقعة على دائرة الوحدة وعمليات الضرب العقدي. وتُوجد هذه النتائج بين المنظورين الجبري والتحليلي للبنى السحرية، كما توفر إنشاءات صريحة تفتح آفاقاً واتجاهات جديدة للبحث في مجالي الجبر والرياضيات المتقطعة.

الكلمات الدالة: المربع السحري، الزمرة السحرية، الزمرة الإبدالية، الزمرة غير الإبدالية، المربع السحري المركب.

1. Introduction

Magic squares are classical objects known for their symmetry and elegance, consisting of $n \times n$ grids of integers in which the sum of each row, column, and diagonal is constant (Andrews, 2004; Pickover, 2002). Beyond integers, mathematicians have explored arrays whose entries are elements of a finite group, leading to the concept of n -magic groups (Cichacz & Fronček, 2026; Gorodetsky, 2024; Trenkler, 2011; Boyer, 1970). In these groups, n^2 distinct elements can be arranged so that row, column, and diagonal products (or sums for abelian groups) are equal. Canonical examples include abelian groups like $Z_n \times Z_n$, where elements can be arranged to sum up to the identity, and non-abelian constructions like $S_3 \times Z_3$, showing that magic balance can extend to more complex operations. The direct product criterion allows generating larger n -magic groups from smaller ones, providing a systematic method to construct diverse examples. Finitely generated abelian 3-magic groups are elegantly characterized: a 3-magic matrix exists if the elements a , b , & c satisfy $(a+b+c = 0)$ and generate a sufficiently large subgroup (Dummit & Foote, 2004; Serre, 1977). This gives a practical framework for building arrays and understanding the algebraic constraints behind them. We also consider complex magic squares, where entries are roots of unity. These arrays are multiplicative relationships that connect magic squares to polynomial roots on the unit circle (Gorodetsky, 2024; Katsura et al., 2022).



2. Materials & Methods

This study explores a new way to look at magic squares—by building them inside mathematical groups instead of just with numbers. We define what it means for a group to be n -magic and show how to construct such arrays for both abelian (commutative) and non-abelian (non-commutative) groups. Using algebraic reasoning, we find consistent “magic” patterns that work across different group types. A general rule, called the direct product criterion, lets us build larger examples from smaller ones. Finally, we connect these ideas to complex numbers and roots of unity, showing how magic group links naturally to patterns in algebra.

3. Definitions and Background

Definition 3.1: An array of integers $\{1, 2, \dots, n^2\}$ that has the constant equal to the sum of each row, column, and both main diagonals is called an n -magic square such that;

$$M = \frac{n(n^2 + 1)}{2}$$

The generalization of the above definition has developed to cover any binary operation (instead of sum) on a group.

Definition 3.2: Let G be a finite group (written multiplicatively, or additively when abelian). The group G is called n -magic if there exist distinct elements $g_{ij} \in G$ arranged in an $n \times n$ array such that

$$\prod_{j=1}^n g_{ij} = \prod_{i=1}^n g_{ij} = \prod_{i=1}^n g_{ii} = \prod_{i=1}^n g_{i,n+1-i} = M$$

for a common element $M \in G$. In the additive case, products are replaced by sums.

4. Direct Product Criterion

This section provides a practical method to generate new n -magic groups systematically; this approach can be applied to both abelian and non-abelian groups, as illustrated in the following examples.



Theorem 4.1: $A \times B$ is km -magic group for any finite groups A and B of order k and m .

Proof. Use the coordinate-wise product formula:

$$P_{S_3 \times \mathbb{Z}_3} = (P_{S_3}^{|\mathbb{Z}_3|}, P_{\mathbb{Z}_3}^{|S_3|})$$

For the factors, note that;

$$P_{S_3} = e \text{ since } Z(S_3) = \{e\}, P_{\mathbb{Z}_3} = e \text{ because } \mathbb{Z}_3 \text{ has odd order and no involutions.}$$

Therefore;

$$P_{S_3 \times \mathbb{Z}_3} = (e^3, e^6) = (e, e)$$

For any $g \in S_3 \times \mathbb{Z}_3$, the product of the elements of the translate $gG = \{gx : x \in G\}$ is

$$\prod_{x \in G} (gx) = g^{18} P_G = (e, e),$$

since $|G| = 18$ and $g^{18} = e$. Hence each row or column of the table multiplies to the identity, and the group is 18 -magic.

Example (1): Let $A = \mathbb{Z}_3$ and $B = \mathbb{Z}_3$. Both of A and B are abelian groups and $|A| = |B| = 3$ divides $n = 3$, so the hypotheses of the theorem hold. The table of elements of $A \times B = \mathbb{Z}_3 \times \mathbb{Z}_3$ is

	$j = 0$	$j = 1$	$j = 2$
$i = 0$	(0,0)	(0,1)	(0,2)
$i = 1$	(1,0)	(1,1)	(1,2)
$i = 2$	(2,0)	(2,1)	(2,2)

Note that, $M = (0,0)$. Hence $\mathbb{Z}_3 \times \mathbb{Z}_3$ is 3-magic.

Similarly, we may construct $Z_4 \times Z_2$ and $n = 8$,

$$Z_4 \times Z_2 = \{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (3,1)\}$$

where;

$$(a, b) + (c, d) = (a + c \bmod 4, b + d \bmod 2)$$

The order is $4 \times 2 = 8$, so this group is expected to be 8 -magic.

Corollary 4.2: In fact, every abelian group of the form $\mathbb{Z}_n \times \mathbb{Z}_m$ is nm -magic.

Example (2): Consider the non-abelian group $G = S_3 \times \mathbb{Z}_3$.



The operation defined on G is $(\sigma, i) \cdot (\tau, j) = (\sigma\tau, i + j \bmod 3)$ where $i, j \in \mathbb{Z}_3$. Since

$$P_{S_3 \times \mathbb{Z}_3} = \prod_{(s,z) \in S_3 \times \mathbb{Z}_3} (s, z) = (e_{S_3}, 0)$$

Hence, $S_3 \times \mathbb{Z}_3$ is 18-magic with magic constant equal to the identity.

5. Characterization of Finitely Generated Abelian 3-Magic Groups

The following result provides a sufficient condition that has been indirectly deduced from Bowerman et al. (2023) practically and easily.

5.1. Proposition (Sufficient conditions for a 3 -magic array)

Proposition 5.1: Let G be an abelian group (written additively). Suppose $a, b \in G$ and let $H = \langle a, b \rangle$. Each of the following hypotheses implies that G admits a 3-magic array (with magic sum 0) whose nine entries lie in H :

1. H contains a subgroup isomorphic to $Z_3 \times Z_3$ (equivalently, H has a subgroup of order 9 of exponent 3); or
2. H contains a cyclic subgroup of order 9 (equivalently, $Z_9 \leq H$).

In particular, if H has a subgroup of order divisible by 9 that is either $Z_3 \times Z_3$ or Z_9 , then G admits a 3 -magic array with magic sum 0 whose entries lie entirely in that subgroup.

Proof: We give explicit constructions in the two cases and verify the magic property.

Case (A): $Z_3 \times Z_3 \leq H$. Choose an isomorphism $Z_3 \times Z_3 \cong \langle u \rangle \times \langle v \rangle \leq H$ where $3u = 3v = 0$ and the 9 distinct elements are $\{iu + jv : i, j \in \{0, 1, 2\}\}$.

Place these nine elements in the 3×3 array indexed by i (row) and j (column) as follows:

$$X = [x_{ij}]_{i,j=0}^2, x_{ij} = iu + jv$$

Compute the sums of the rows (all computations are in the subgroup and coordinates mod 3) :

$$\sum_{j=0}^2 x_{ij} = \sum_{j=0}^2 (iu + jv) = 3iu + (0 + 1 + 2)v = 0 + 3v \equiv 0$$



since $3iu = 0$ and $3v = 0$. Thus, every row sum is 0. Similarly, for each fixed column index j ,

$$\sum_{i=0}^2 x_{ij} = \sum_{i=0}^2 (iu + jv) = (0 + 1 + 2)u + 3jv = 3u + 0 \equiv 0.$$

Finally, the main diagonal sum is

$x_{00} + x_{11} + x_{22} = (0u + 0v) + (1u + 1v) + (2u + 2v) = 3u + 3v \equiv 0$
and the anti-diagonal sum $(0,2), (1,1), (2,0)$ equals

$$(0u + 2v) + (1u + 1v) + (2u + 0v) = 3u + 3v \equiv 0$$

Thus $M = 0$, and the entries are the 9 distinct elements of the subgroup $Z_3 \times Z_3$. So this produces a 3-magic with magic sum 0.

Case (B): $Z_9 \leq H$. Let $g \in H$ be an element of order 9, so $\langle g \rangle \cong Z_9$ and its elements are $\{0, g, 2g, \dots, 8g\}$. We pick the nine elements

$$\{0g, 1g, 2g, 3g, 4g, 5g, 6g, 7g, 8g\}$$

and arrange them in three rows whose coordinate-sums are multiples of $9g = 0$ (i.e. zero in H). A simple partition of $\{0, \dots, 8\}$ into three triples with sums divisible by 9 is:

$$\{0, 1, 8\}, \{2, 3, 4\}, \{5, 6, 7\}$$

because

$$0 + 1 + 8 = 9, 2 + 3 + 4 = 9, 5 + 6 + 7 = 18$$

and each of $9g$ and $18g$ equals 0. Thus, take the rows

$$R_1 = (0g, 1g, 8g), R_2 = (2g, 3g, 4g), R_3 = (5g, 6g, 7g)$$

We check columns and diagonals:

Column sums: column 1 sum $= 0g + 2g + 5g = 7g$; column 2 sum $= 1g + 3g + 6g = 10g = g$; column 3 sum $= 8g + 4g + 7g = 19g = g$. These three are not equal yet, so we must permute rows/columns to force column sums to be zero as well.

A convenient symmetric arrangement that works is to place the triples in the 3×3 matrix as;

$$\begin{pmatrix} 0g & 1g & 8g \\ 3g & 4g & 2g \\ 6g & 7g & 5g \end{pmatrix}$$



Thus; case B is constructive: because Z_9 is cyclic of order 9, one can explicitly choose nine distinct multiples of the generator and place them so $M = 0(\text{mod}9)$.

Remark on the general claim $|H| \geq 9$. The mere inequality $|H| \geq 9$ is not sufficient without further structural information: for instance if $|H| = 10$ but $H \cong Z_{10}$ (and $9 \nmid 10$ and H contains neither Z_9 nor $Z_3 \times Z_3$), the simple constructions above do not immediately apply and one must work harder to produce nine elements whose sums align to zero. The two cases above (existence of $Z_3 \times Z_3$ or of Z_9) are natural and commonly sufficient conditions that are easy to apply and verify; they cover many groups of interest (e.g. any subgroup whose order is a multiple of 9 contains one of these as a subgroup). The full necessary-and-sufficient classification for all finitely generated abelian groups is more intricate and is given in Bowerman et al. (2023).

Lemma 5.2: A finitely generated abelian group G admits a 3 -magic 3×3 array (with nine distinct entries and magic constant the identity) if and only if G contains a subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof: (Sufficiency). Suppose G contains a subgroup $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Write $H = \langle u, v \rangle$ with $3u = 3v = 0$. Define a 3×3 array by;

$$M_{i,j} = iu + jv, i, j \in \{0,1,2\}$$

The nine elements are distinct since $|H| = 9$. Each row and each column sums to;

$$\sum_{j=0}^2 (iu + jv) = 3iu + (0 + 1 + 2)v = 0$$

So, the magic constant is the identity. Hence G admits a 3-magic array. (Necessity) Conversely, assume G has a 3 -magic 3×3 array $x_{i,j}$ with nine distinct entries and row/column sums equal to 0. Let the first row be (a, b, c) , so $a + b + c = 0$. Define

$$u := x_{1,0} - x_{0,0}, v := x_{0,1} - x_{0,0}$$

Using the row and column sum identities one checks that;

$$x_{i,j} = a + iu + jv, i, j \in \{0,1,2\}$$



Distinctness of entries implies that $\{iu + jv: i, j \in \{0, 1, 2\}\}$ are all distinct, so $|\langle u, v \rangle| \geq 9$.

The 3-magic conditions force $3u = 3v = 0$, hence $\langle u, v \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Therefore; G contains a subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Example (3): Let $H = \langle u, v \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ such that $3u = 3v = 0$. The 3×3 array can be structured for 3-magic with $M = 0$ as:

$$\begin{pmatrix} 0 & v & 2v \\ u & u + v & u + 2v \\ 2u & 2u + v & 2u + 2v \end{pmatrix}$$

All nine entries are distinct, and the magic constant is the identity 0 of H .

6. Complex n-Magic Group

The concept of a magic group on the complex plane, particularly It is a rich and elegant framework, based on the roots of unity. for constructing n -magic arrays. These arrays not only generalize the classical notion of magic squares to a multiplicative setting but also allow the Algebraic and geometric study of the structure of the roots of polynomials on the unit circle. The interplay between group-theoretic properties and Complex multiplication produces some interesting patterns, symmetries, and many applications

Definition 6.1: An $n \times n$ matrix $M = [z_{ij}]_{1 \leq i, j \leq n}$, $z_{ij} \in \mathbb{C}$ is called a complex magic square if there exists $S \in \mathbb{C}$ such that;

$$\sum_{j=1}^n z_{ij} = \sum_{i=1}^n z_{ij} = \sum_{i=1}^n z_{ii} = \sum_{i=1}^n z_{i, n+1-i} = S.$$

The complex number S is the magic sum.

Example (4): Let $\omega = e^{2\pi i/3}$. Then;

$$M = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}$$

is a complex magic square with magic sum: $S = 1 + \omega + \omega^2 = 0$



Every row, column, and diagonal sums to 0. This example encodes the additive relations among the cube roots of unity and illustrates the connection between complex magic squares and character tables of cyclic groups.

Example (5): Let $G = U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$. For odd n , define $g_{i,j} = e^{2\pi i(i+j)/n}$. Then;

$$\prod_{j=1}^n g_{i,j} = e^{2\pi i(in+n(n+1)/2)/n} = e^{\pi i(n+1)} = 1$$

So, every row, column and diagonal product equals 1. Thus $U(1)$ is an n -magic group for odd n .

Or we could let $\omega = e^{2\pi i/n}$ and $G = \langle \omega \rangle = \{\omega^k : k = 0, \dots, n-1\} \subset \mathbb{C}$. Define $g_{i,j} = \omega^{i+j}$. Then;

$$\prod_{j=1}^n g_{i,j} = \omega^{n(n+1)/2}, \prod_{i=1}^n g_{i,j} = \omega^{n(n+1)/2}$$

and both diagonal products yield the same element. Hence G is a complex n -magic group with magic product $M = \omega^{n(n+1)/2}$.

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