PARETO-S DISTRIBUTION

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ABSTRACT

In this article, the Pareto distribution will be converted to a new distribution using the DUS method, with some statistical properties and estimation parameters of the new distribution being studied. It's named Pareto-S distribution. Some statistical properties of the Pareto-S distribution are obtained such as, median, mode, variance, moments, moment generating function, characteristic function, reliability, hazard function and Rényi entropy. Pareto-S Parameter Estimation by maximum likelihood method and Newton-Raphson procedure are addressed. Finally, the estimation procedures to different simulated Pareto-S data sets each with pre-specified parameter values and varying sample sizes are applied, and the results of Newton-Raphson procedure compared with the results of the ML method, the results show that the ML seem to perform much better than the Newton-Raphson method for the small sample sizes and the Newton-Raphson method seem to perform much better than the ML for the large sample sizes.

Keywords: DUS Method; Reliability Function, Hazard Function; Maximum Likelihood Method; Newton-Raphson Procedure; Pareto Distribution; Hazard Function; Rényi Entropy.

توزيع باريتو-أس

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الملخص

في هذا المقال سيتم تحويل توزيع باريتو إلى توزيع جديد باستخدام طريقة DUS، مع دراسة بعض الخصائص الإحصائية ومعايير التقدير للتوزيع الجديد والذي يسمى توزيع باريتو -أس. تم الحصول على بعض الخصائص الإحصائية لتوزيع باريتو –أس مثل الوسيط، المنوال، التباين، العزوم، دالة توليد العزوم، الدالة المميزة، الموثوقية، دالة الخطر وإنتروبيا ريني. يتم تناول تقدير معالم توزيع باريتو –أس بواسطة طريقة الاحتمالية القصوى وإجراء نيوتن-رافسون، أخيرًا تم تطبيق إجراءات التقدير لمجموعات بيانات باريتو -أس المختلفة لكل منها قيم معلمات محددة مسبقًا وأحجام عينات مختلفة عن طريق المحاكاة، ومقارنة نتائج إجراء نيوتن-رافسون مع نتائج طريقة الأرجحية القصوى، أظهرت النتائج أن طريقة الأرجحية لها أداء أفضل بكثير من طريقة نيوتن-رافسون بالنسبة لأحجام العينات الصغيرة، بينما طريقة نيوتن-رافسون تؤدي أداءً أفضل بكثير من طريقة تعلم الآلة بالنسبة لأحجام العينات الكبيرة.

الكلمات الرئيسية: طريقة DUS، دالة الموثوقية، دالة المخاطر، طريقة الاحتمالية القصوى، إجراء نيوتن-رافسون، توزيع باريتو، ريني إنتروبيا.

1. INTRODUCTION

Pareto distribution is a probability distribution that has many applications, and is widely used in applications of reliability theory because it is one of the failure distributions. Also, it is used to describe the income distribution and the extreme behavior of the value of loss in the economic field. In fields economic, called this distribution (Pareto distribution) after the economist professor Flevry Barreto (1848-1923). The Pareto distribution is a power-law probability distribution that is used in description of social, quality control, scientific, geophysical, actuarial, and many other types of observable phenomena. Originally applied to describing the distribution of wealth in a society, fitting the trend that a large portion of wealth is held by a small fraction of the population, (Vilfredo, 1898). There are many forms of the Pareto distribution, such as Pareto's law of the second type (Lomax's law), Pearson's law of the sixth type, Pareto law of the third type, Pareto law of the fourth type, generalized Pareto distribution and so on, (Vilfredo, 1898).

It has many applications in science, life testing, and so on, for more details see (Lee and Kim, 2018). In addition, there are many authors applied Pareto distribution to another models, such as Levy and Levy (2003) used Pareto distribution for investigation of wealth in society. Farshchian (2010) applied Pareto distribution to model sea clutter intensity returns. Gómez-Déniz, and Calderín-Ojeda (2015) developed ArcTan Pareto distribution and they applied it to model insurance and population size data. Malik and Ahmad (2017) introduced two-parameter alpha power Rayleigh distribution. Omekam, et al. (2022) extended Pareto type-I distribution to derive some functions using five parameter induction methods.

2.PARETO-S DISTRIBUTION

The Pareto distribution with pdf (1) and cdf(2) was transformed into a new one using the DUS method which is proposed by Dinesh Kumar, *et. al.* (2015), and it is named Pareto-S distribution.

$$f^*(x) = \frac{\alpha\beta^{\alpha}}{x^{(\alpha+1)}} \quad x \geq \beta \;, \alpha, \beta > 0 \;\; (1)$$

$$F^*(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha} \quad x \geq \beta \;, \alpha, \beta > 0 \;\; (2)$$

In light of the DUS method, the pdf and the cdf of the Pareto-S distribution are defined as follows:

$$f(x) = \frac{1}{e-1} f^*(x) \cdot e^{F^*(x)} = \frac{\alpha\beta^{\alpha}}{(e-1)} x^{-(\alpha+1)} \cdot e^{1-\left(\frac{\beta}{x}\right)^{\alpha}} \;, \quad x \geq \beta \;, \alpha, \beta > 0 \qquad (3)$$

$$F(x) = \frac{e}{(1-e)}\left(0.368 - e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right). \qquad x \geq \beta \;, \alpha, \beta > 0 \qquad (4)$$

Here α the shape parameter and β the location parameter. The function of the new formula for the Pareto-S distribution is a probability density function.

3. SOME STATISTICAL PROPERTIES OF PARETO-S DISTRIBUTION

Moments: $\mu'_r = \frac{e\beta^r}{(e-1)} \gamma\left(1 - \frac{r}{\alpha}, 1\right) \;, \alpha > r \;, \quad r = 1,2,...$

Moment Generating Function: $M_x(t) = \frac{e}{(e-1)} \sum_{j=0}^{\infty} \frac{(t\beta)^j}{j!} \; \gamma\left(1 - \frac{j}{\alpha}, 1\right) \;, \;\; \forall\, \alpha > j$

Characteristic Function: $Q_x(t) = \frac{e}{(e-1)} \sum_{j=0}^{\infty} \frac{(it\beta)^j}{j!} \ \gamma(1-\frac{j}{\alpha},1)$

Random Variables: $x_R = \beta\left\{\ln\left(0.368 - \frac{U(e-1)}{e}\right)^{-1}\right\}^{\frac{-1}{\alpha}}$

Median: $M = x_m = \beta\left\{\ln\left(0.368 - \frac{(e-1)}{2e}\right)^{-1}\right\}^{\frac{-1}{\alpha}}$

Mode: $\hat{x} = \beta\left\{\frac{\alpha+1}{\alpha}\right\}^{\frac{-1}{\alpha}}$

Reliability Function: $R(x) = 1 - F(x) = 1 - \frac{e}{(1-e)}\left(0.368 - e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right)$

Hazard Function: $H(x) = \frac{f(x)}{R(x)} = \frac{\alpha\beta^{\alpha}(x)^{-(\alpha+1)} \cdot e^{-\left(\frac{\beta}{x}\right)^{\alpha}}}{1 - e^{-\left(\frac{\beta}{x}\right)^{\alpha}}}$

At different values of the distribution parameters, the graphs shows that, pdf takes different shapes, cdf is concave and extending to the right as well. Also, the reliability function (RF) is convex in shape extending to the right with hazard function (HF) , starting to be increasing function and then decreasing function

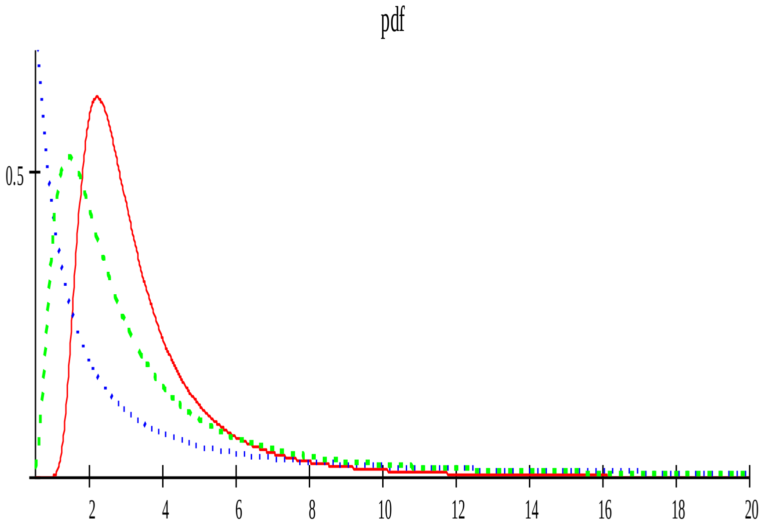


Figure 2.5:The pdf of Pareto-S Distribution

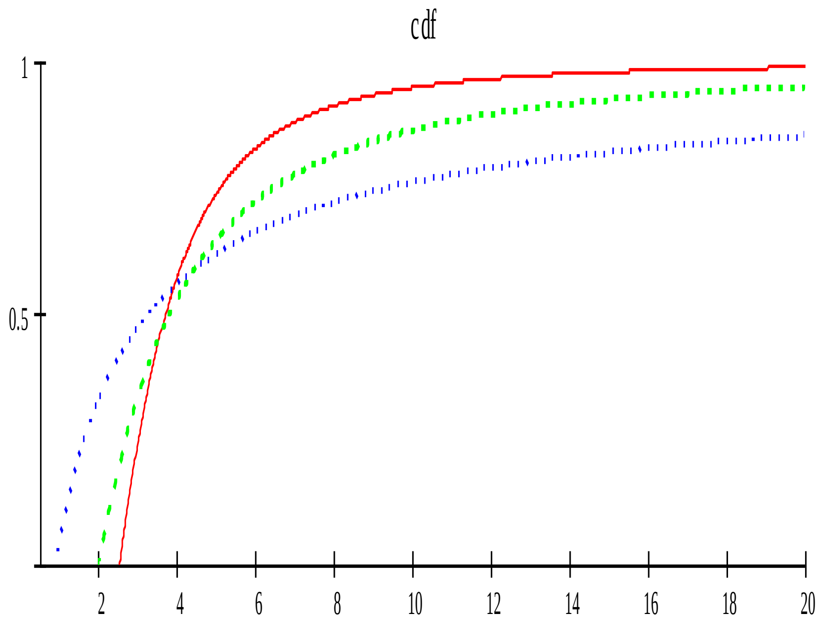


Figure 2.6:The cdf of Pareto-S Distribution

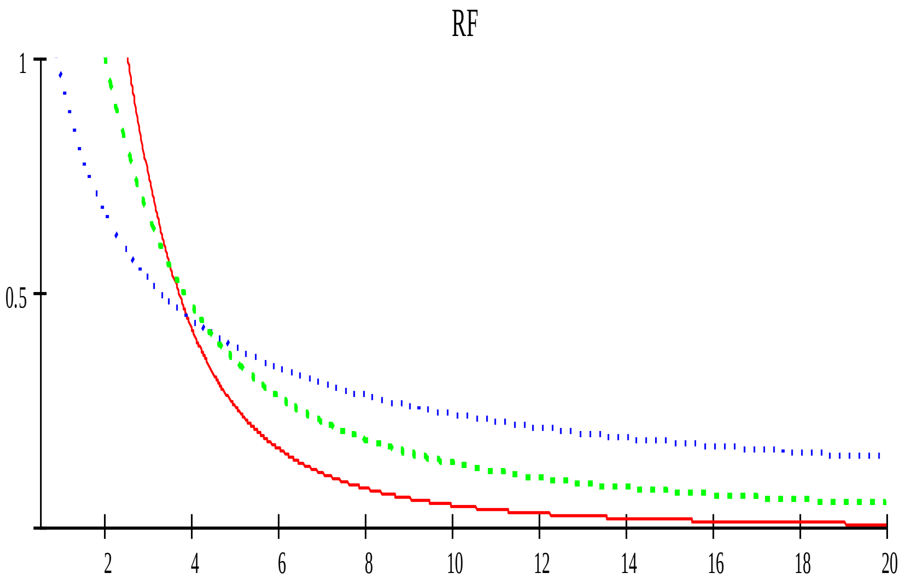


Figure 2.7:The RF of Pareto-S Distribution

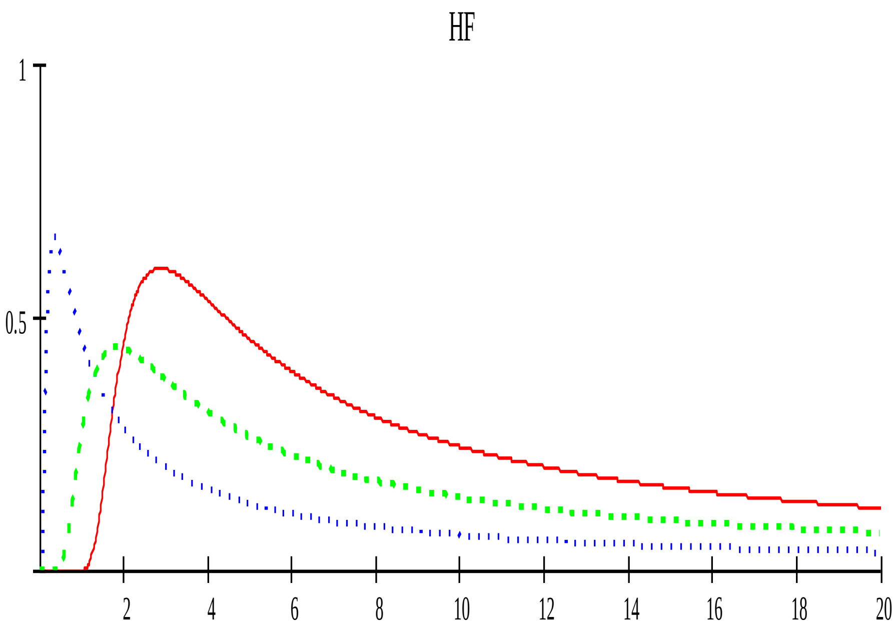


Figure 2.8:The HF of Pareto-S Distribution

4. PARETO-S PARAMETER ESTIMATION

Let $X_1, X_2, ..., X_n$ a iid random sample of size n and the order statistics be $x_{(1)} < x_{(2)} < ... < x_{(n)}$, then the log likelihood function of the Pareto-S with parameter (α) can be obtained as follows:

$$Logl(\alpha) = nLog\left(\frac{e}{(e-1)}\right) + nLog(\alpha) + n\alpha\ Log(\beta) - (\alpha+1)\sum_{i=1}^{n} Log(x_i) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}$$

Therefore,

$$\hat{\alpha} = \frac{n}{-n\ Log(\hat{\beta}) + \sum_{i=1}^{n} Log(x_i) + \sum_{i=1}^{n}\left(\frac{\hat{\beta}}{x_i}\right)^{\hat{\alpha}} Log\left(\frac{\hat{\beta}}{x_i}\right)}$$

Where, $\hat{\beta} = x_{(1)}$ and the ML estimator $(\hat{\alpha})$ for the unknown parameters (α) will be solved numerically within the Newton-Raphson**.** Newton's method is quite applicable for maximum likelihood estimation when the roots of $l'(\alpha) = 0$ do not have a closed form expression, then this problem could be solved numerically within the Newton-Raphson iterated procedure framework as follows:

the Taylor series to solve the probability equation:

$$\frac{\partial Log\ l(\alpha)}{\partial \alpha} = \frac{n}{\hat{\alpha}} + n\ Log(\beta) - \sum_{i=1}^{n} Log(x_i) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\hat{\alpha}} Log\left(\frac{\beta}{x_i}\right) = 0$$

By equating Taylor's equation of first degree to zero, we find

$$\alpha^{(m+1)} - \alpha^{(m)} = \frac{\frac{n}{\alpha^{(m)}} + n\ Log(\hat{\beta}) - \sum_{i=1}^{n} Log(x_i) - \sum_{i=1}^{n}\left(\frac{\hat{\beta}}{x_i}\right)^{\alpha^{(m)}} Log\left(\frac{\hat{\beta}}{x_i}\right)}{\frac{n}{\alpha^{(m)2}} + \sum_{i=1}^{n}\left(\frac{\hat{\beta}}{x_i}\right)^{\alpha^{(m)}}\left(Log\left(\frac{\hat{\beta}}{x_i}\right)\right)^2} \quad (5)$$

given an initial guess α_0, the eq.(5) can then be solved for α, and the iterative solutions will take the form,

$$\alpha^{(m+1)} = \alpha^{(m)} + \frac{\dfrac{n}{\alpha^{(m)}} + n\,Log\left(\hat{\beta}\right) - \sum_{i=1}^{n} Log\left(x_i\right) - \sum_{i=1}^{n}\left(\dfrac{\hat{\beta}}{x_i}\right)^{\alpha^{(m)}} Log\left(\dfrac{\hat{\beta}}{x_i}\right)}{\dfrac{n}{\alpha^{(m)^2}} + \sum_{i=1}^{n}\left(\dfrac{\hat{\beta}}{x_i}\right)^{\alpha^{(m)}}\left(Log\left(\dfrac{\hat{\beta}}{x_i}\right)\right)^2}, \quad m = 1,2,3,\ldots$$

then, the new parameter value $\alpha^{(m+1)}$ is obtained at each iteration (m), the convergence point is a maximum point if $I(\alpha^{(m)}) < 0$, and the iterations continued until, $|\alpha^{(m+1)} - \alpha^{(m)}| < \varepsilon\ , \varepsilon > 0$.

5. A SIMULATION STUDY

To estimate the ML Pareto-S parameter for ten different simulated random samples of sample sizes ranging from 5 to 200 is applied. This is to show the effect of the sample size to the quality of the estimates. The method of ML is to be applied to estimate the parameter of Pareto-S distribution and its mean square error. The same calculations are done for all generated ten random samples. Table (1) presents the estimates of the Pareto-S parameter along with its mean square error. The results show that the estimates of the parameter is a reasonable for all considered random samples of sizes and the mean square errors decreases as the sample size increasing which give an indication that the performance of the ML method is as good as we hope. Also, only a few iterations are required before achieving the convergence criteria as can be seen from Table (1). The estimate values are getting better even for small sample sizes (5, 7, 10) compared to the method of Newton Raphson, see Table(2).

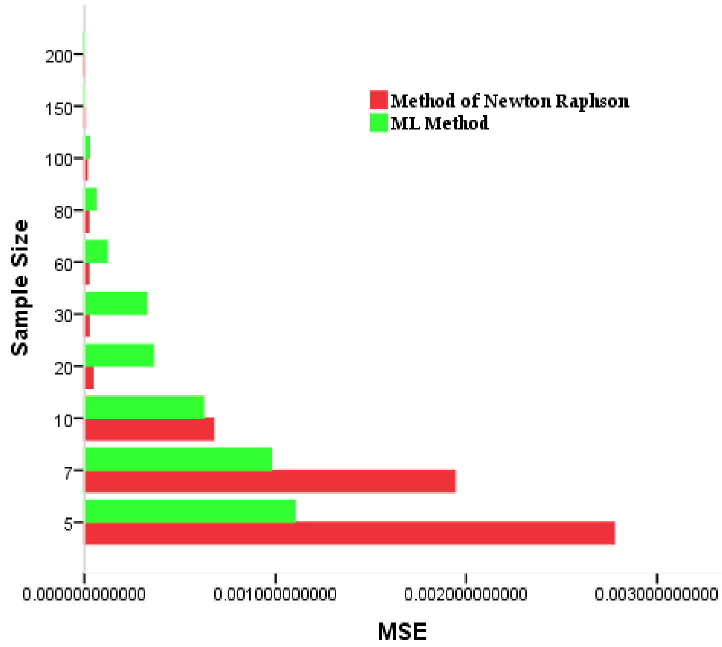
TABLE 1 : ML PARAMETER ESTIMATES

n	α_0	$\hat{\alpha}$	$\hat{\beta}$	MSE($\hat{\alpha}$)	iteration
5	0.31	0.252	0.054	1.10310E-3	3
7	0.35	0.433	1.656	9.78510E-4	7
10	0.31	0.26	0.3116	6.23810E-4	4
20	0.32	0.27	0.09	3.58410E-4	7
30	0.162	0.216	0.036	3.26210E-4	9
60	0.2	0.167	3.58810E-3	1.17810E-4	9
80	0.18	0.161	2.33810E-3	6.01710E-5	6
100	0.2	0.181	7.63410E-3	2.68810E-5	13

The same simulated data sets are to be used again to estimate the unknown parameter of the Pareto-S distribution using Newton-Raphson method. The results are given in Table (2). The number of iterations for each sample is also given and it can be seen that only a few iterations are required before achieving the convergence criteria. From the mean square error point of view, the estimate values here are getting better even for small sample sizes, we can notes that there is a slight improvement when applying the Newton Raphson method especially at the sample size (20, 30, 60, 80, 100, 150, 200) compared with the results of the ML method, see Figure(1). As the estimated mean square error values, for both estimated parameters, are quite smaller.

TABLE 2 : NEWTON RAPHSON PARAMETER ESTIMATES

n	α_0	$\hat{\alpha}$	$\hat{\beta}$	MSE($\hat{\alpha}$)	iteration
5	0.31	0.384	0.077	2.77410E-3	2
7	0.35	0.252	1.156	1.94010E-3	5
10	0.31	0.258	0.311	6.76610E-4	4
20	0.32	0.337	0.09	4.31610E-5	7
30	0.162	0.169	5.49510E-3	2.47210E-5	2
60	0.2	0.215	3.58810E-3	2.41210E-5	9
80	0.18	0.192	2.33810E-3	2.37310E-5	6
100	0.2	0.214	7.63410E-3	1.46110E-5	13
150	0.15	0.154	7.00410E-3	3.67010E-7	5
200	0.15	0.153	1.78210E-3	1.76110E-7	6



Figure(1): MSEs of ML Method and Newton Raphson Method

6. CONCLUSION

Throughout this article many important points and useful results are obtained and the main features that for Pareto distribution the moment generating function is only defined for non-positive values $t \leq 0$, but for Pareto-S is defined for all positive values of t . Also, to obtain the Pareto-S estimate, the ML seem to perform much better than the Newton-Raphson method especially for small sample sizes (5, 7, 10), and the Newton-Raphson method seem to perform much better than the ML especially at the sample size (20, 30, 60, 80, 100).The number of iterations required when applying the ML and the Newton-Raphson method is very small (2 to 13) and this is partly due to the good choice of the initial values used.

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